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Discretization Method for Semi-Definite Programming

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Abstract—Semi-definite programs are convex optimization problems arising in a wide variety of applications and the extension of linear programming. Most methods for linear programming have been generalized to semi-definite programs. This paper discusses the discretization method in semi-definite programming. The convergence and the convergent rate of error between the optimal value of the semi-definite programming problems and the optimal value of the discretized problems are obtained. An approximately optimal division is given under certain assumptions. With the significance of the convergence property, the duality result in semi-definite programs is proved in a simple way which is different from the other common proofs.

Keywords—Semi-definite programming, Duality, Semi-infinite programming, Optimal division.

1. INTRODUCTION

Consider the following *semi-definite programming problem* (SDP):

$$\begin{aligned} &\min && c^\top x, \\ &\text{subject to} && F(x) \succeq 0, \end{aligned} \tag{P}$$

where $F(x) := F_0 + \sum_{i=1}^m x_i F_i$, $c \in R^m$, $F_i \in R^{n\times n}$, $i = 0, 1, \ldots, m$ are symmetric and linearly independent. The inequal "$\succeq$" denotes the Löwner partial order, i.e., $F(x) \succeq 0$ denotes that $y^\top F(x) y \geq 0$ for all $y \in R^n$. Obviously, the feasible set of (P) is convex.

Semi-definite programs are an important class of convex optimization which directly arises in a number of important applications such as control theory, combinatorial optimization, matrix theory and etc. Many convex optimization problems, e.g., linear programming and (convex) quadratically constrained quadratic programming can be cast as semi-definite programs. So semi-definite programming offers a unified way to study the properties and derive algorithms for a wide variety of convex optimization problems. Recently, interior-point methods for linear programming have been generalized to semi-definite programs [1] such as primal-dual interior-point method and potential reduction method. As in linear programming, these methods have

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polynomial worst-case complexity and perform very well in practice. We can also view the semi-definite program (P) as a semi-infinite LP. So, in this paper we consider an extension of the discretization solution method, which performs efficiently in semi-infinite programming.

The (SDP) may be written equivalently as the following semi-infinite problem:

$$\min \left\{ c^\top x \mid x \in X \right\}, \tag{P$'$}$$

where $X = \{x \mid y^\top F(x) y \geq 0,\ y \in Y\}$, and $Y = \{y \mid \alpha \leq \|y\|^2 \leq 1\}$ $(0 < \alpha < 1)$ is a compact set for any given α.

A common approach for solving semi-infinite programming problem is the discretization method (see [2,3]). For (P$'$), one chooses a finite grid of $Y_d = \{y_1, y_2, \dots, y_l\} \subset Y$, and solves the following discretized problem:

$$\begin{array}{ll} \min & c^\top x \\ \text{subject to} & x \in X(Y_d) = \left\{ x \in R^m \mid y^\top F(x) y \geq 0,\ y \in Y_d \right\} \end{array} \tag{P$_d$}$$

In this paper, we apply the discretization method in semi-infinite program to (SDP).

The organization of this paper is as follows. In Section 2, some preliminary knowledge is recalled. In Section 3, based on [2], we obtain the convergence rate of error between the optimal value of the semi-definite programming problem and the optimal value of the discretized problem and a simple proof of the duality result in semi-definite programming is given, which differs from previous ones. Section 4 discusses how to get optimal division Y_d for a given tolerance.

In this paper, d denotes the Hausdorff distance between Y and Y_d, i.e.,

$$d := \operatorname{dist}(Y_d, Y) = \max_{y \in Y} \min_{\hat{y} \in Y_d} \|\hat{y} - y\|,$$

where $\|\cdot\|$ denotes the Euclidean norm. d measures the fineness of the mesh grid Y_d.

2. PREPARATIONS

In this section, we review some results in matrix theory and in semi-infinite programming whose special case is (P$'$).

LEMMA 1. *(See [4].) Let A and B are symmetric $n \times n$ matrices. If $A \succeq 0, B \succeq 0$, then*

(a) $\operatorname{Tr} AB \geq 0$, *and*
(b) $\operatorname{Tr} AB = 0$,

if and only if $AB = 0$.

Now, consider the following semi-infinite programming problem:

$$\begin{array}{ll} \min & f(x), \\ \text{subject to} & x \in X = \{x \in R^m,\ g(x,y) \leq 0,\ y \in Y\}, \end{array} \tag{SIP}$$

where f, g are given linear real-valued functions and $Y \subset R^n$ is a given compact infinite index set.

With the discretization method to solve (SIP), we choose a finite gird Y_d of Y, $Y_d \subset Y$, and get the discretized problem of (SIP):

$$\begin{array}{ll} \min & f(x), \\ \text{subject to} & x \in X(Y_d) = \{x \in R^m,\ g(x,y) \leq 0,\ y \in Y_d\}, \end{array} \tag{SIP(Y_d)}$$

where $d := \operatorname{dist}(Y_d, Y)$.

LEMMA 2. *(See [5].) Consider the linear (SIP) with $f(x) = \lambda^\top x$, $g(x,y) = a^\top(y)x - b(y)$. Assume that all P-level sets (cf., Definition 1 in Section 3) are bounded. Then, for every $\epsilon > 0$, there exists $\delta_\epsilon > 0$, such that for $Y_{\hat{d}} \subset Y$, $d(Y_{\hat{d}}) < \delta_\epsilon$, it is the case that SIP($P_{\hat{d}}$) is solvable and for every solution $x_{\hat{d}}$, there exists a solution x^* of (SIP), such that $\|x_{\hat{d}} - x^*\| < \epsilon$.*

Now let $\bar{x}$ be a local solution of (SIP), and for each d, let x_d be a local solution of SIP(Y_d), such that $x_d \longrightarrow \bar{x}$ as $d \longrightarrow 0$.

We recall three assumptions introduced in [2].

(A_1). *Let the following hold.*

(a) *There is a neighborhood U of $\bar{x}$, such that the function $D_y^2 g(x,y)$ is continuous on $U \times Y$.*

(b) *The index set $Y \in R^n$ is compact, nonempty and explicitly given as the solution set of inequalities*

$$Y = \{y \in R^n \mid v_i(y) \le 0,\ i \in I\}, \tag{2.1}$$

where I is a finite index set and $v_i \in C^2(y)$.

(c) *The Mangasarian-Fromovitz Constraint Qualification (MFCQ) holds for Y: for any $\bar{y} \in Y$ with the active set $I(\bar{y}) := \{i \in I \mid v_i(\bar{y}) = 0\}$ there exists a vector $\bar{\eta} = \eta(\bar{y})$, such that*

$$Dv_i(\bar{y})\,\bar{\eta} < 0, \qquad i \in I(\bar{y}). \tag{2.2}$$

Let us fix a subset I_*, such that there exists a point $y_* \in Y$ with $I_* = I(y_*)$. There are finitely many such index sets I_*, say $I_1, I_2, \dots, I_k$. Define the set

$$S_j := \{y \in Y \mid v_i(y) = 0,\ i \in I_j\}, \qquad j = 1, \dots, k. \tag{2.3}$$

(A_2). *Suppose Y satisfies (A_1) and the sets $S_j, S_j \cap Y \ne \emptyset$, $j = 1, \dots, k$. Let the grids Y_d be chosen, such that*

$$\max_{y \in S_j} \min_{\hat{y} \in Y_d \cap S_j} \|y - \hat{y}\| \le d, \qquad \text{for all } j = 1, \dots, k, \text{ and all } d. \tag{2.4}$$

(A_3). *Assume that $D_x g(x,y)$ is continuous on $U \times Y$, where U is a neighborhood of $\bar{x}$. Let moreover MFCQ be valid at the local minimizer $\bar{x}$ of (SIP), i.e., there exists a vector $\xi \in R^m$, such that*

$$D_x g(\bar{x}, \bar{y})\,\xi \le -1, \qquad \text{for all } \bar{y} \in Y_0(\bar{x}), \tag{2.5}$$

with $Y_0(\bar{x}) = \{y \in Y\bar{g}(\bar{x}, y) = 0\}$, the active set index at $\bar{x}$.

LEMMA 3. *(See [2].) Let (A_3) hold and let $f(x)$ be Lipschitz continuous near $\bar{x}$. Then the following is true.*

(a) *There is some $\gamma_1 > 0$, such that for all d small enough,*

$$0 \le f(\bar{x}) - f(x_d) \le \gamma_1 \bar{d}. \tag{2.6}$$

(b) *If in addition, (A_1) and (A_2) are satisfied for Y, Y_d, then with some $\gamma_2 > 0$ the inequality*

$$0 \le f(\bar{x}) - f(x_d) \le \gamma_2 d^2 \tag{2.7}$$

is valid for small d.

3. CONVERGENCE AND DUALITY

3.1. Convergence and Convergent Rate

In this section, we continue to explore the discretization method in semi-definite programming and the convergent rate of error between the optimal value of the semi-definite programming problem and the optimal value of the discretized problem.

First, we give the following denotations and definitions:

$$X^P(Y) = \left\{x \mid y^\top F(x) y \geq 0, \text{ for any } y \in Y\right\} \subset R^n.$$

It can be seen that $X^P(Y)$ is convex. Let $Y_d \subset Y$, $|Y_d| < \infty$,

$$X^P(Y_d) = \left\{x \mid y^\top F(x) y \geq 0, \text{ for any } y \in Y_d\right\}.$$

DEFINITION 1. *P-level sets (for level κ) is defined by*

$$L_{\succeq}\left(X^P, c, \kappa\right) = \left\{x \in X^P(Y) \mid c^\top x \leq \kappa\right\}.$$

DEFINITION 2. *Problem (P) is strictly feasible if there exists an x with $F(x) \succ 0$.*

DEFINITION 3. *Problem (D) is strictly feasible if there exists a Z with $Z = Z^\top \succ 0$, $\operatorname{Tr} F_i Z = c_i$, $i = 1, \ldots, m$.*

LEMMA 4. *Considering problem (P), if (P) and (D) are both strictly feasible, we have all the P-level sets are bounded.*

PROOF. Since (P) is strictly feasible, there is an $\tilde{x}$ with

$$y^\top F(\tilde{x})\, y > 0, \qquad \text{for all } y \in Y.$$

If the statement is false, we must have a P-level set denoted by

$$L_{\succeq}\left(X^P, c, \kappa\right) = \left\{x \in X^P \mid c^\top x \leq \kappa\right\}$$

is unbounded. Then there exists a ray $\{\tilde{x} + th\}_{t>0}$, such that

$$\{\tilde{x} + th\}_{t>0} \subset L_{\succeq}\left(X^P, c, \kappa\right) = \left\{x \in X^P \mid c^\top x \leq \kappa\right\} \subset X^P(Y)$$

with some $h \in R^m$ and $h \neq 0$. So we have for any $t > 0$

$$y^\top F_0 y + \sum_{i=1}^m y^\top \tilde{x}_i F_i y + t \sum_{i=1}^m y^\top h_i F_i y \geq 0,$$

for all $y \in Y$. That is

$$F_0 + \sum_{i=1}^m \tilde{x}_i F_i + t \left(\sum_{i=1}^m h_i F_i\right) \succeq 0$$

for any $t > 0$. So we conclude that

$$\sum_{i=1}^m h_i F_i \succeq 0.$$

With the condition that (D) is strictly feasible, we have some $Z = Z^\top \succ 0$, such that $\operatorname{Tr} F_i Z = c_i$, $i = 1, \ldots, m$. By Lemma 1, we have

$$\sum_{i=1}^m h_i \operatorname{Tr}(F_i Z) = \operatorname{Tr}\left(\sum_{i=1}^m h_i F_i\right) Z = c^\top h \geq 0.$$

If $c^\top h = 0$, then $(\sum_{i=1}^m h_i F_i) Z = 0$. Since $(\sum_{i=1}^m h_i F_i) \succeq 0$, $Z \succ 0$, we can conclude $\sum_{i=1}^m h_i F_i = 0$ from Lemma 1(b). It contradicts the fact that $F_i, i = 1, \ldots, m$ are linearly independent. If $c^\top h > 0$, then $c^\top (x + th) \longrightarrow +\infty$ as $t \longrightarrow +\infty$. It contradicts the fact that

$$\{x + th\}_{t>0} \subset L_{\succeq}\left(X^P, c, \kappa\right) = \left\{x \in X^P \mid c^\top x \leq \kappa\right\}.$$

Hence, the statement is true. ∎

THEOREM 1. *Considering the problem (P), for every* $\epsilon > 0$, *there exists* $\delta_\epsilon > 0$, *such that for* $Y_d \subset Y$, $d(Y_d) < \delta_\epsilon$, *it is the case that* (P_d) *is solvable and for every solution* x_d, *there exists a solution* x^* *of (P), such that* $\|x_d - x^*\| < \epsilon$.

PROOF. Notice that problem (P) is equivalent to problem (P′), and with Lemmas 2 and 4, the proposition is clear.

With the above result, we can select $Y_d \subset Y$ with $d \longrightarrow 0$ and then $x_d \longrightarrow x^*$ as $d \longrightarrow 0$, where x_d is the solution of (P_d) and x^* is the solution of (P). Since (P) and (P_d) are convex programs, x^* and x_d are the global solution of (P) and (P_d), respectively.

Set $S_1 = \{y \mid \|y\|^2 = \alpha\}$, $S_2 = \{y \mid \|y\|^2 = 1\}$. ∎

THEOREM 2. *Suppose* $Y_d \subset Y$ *be chosen, such that* S_j, $S_j \cap Y \neq \emptyset$, $j = 1, 2$ *and*

$$\max_{y \in S_j} \min_{\hat{y} \in Y_d \cap S_j} \|y - \hat{y}\| \le d \qquad \text{for } j = 1, 2 \text{ and all } d. \tag{3.1}$$

If (P) and (D) are strictly feasible, there is some $\gamma_3 > 0$, *such that for d small enough,*

$$0 \le c^\top x^* - c^\top x_d \le \gamma_3 d^2. \tag{3.2}$$

PROOF. First, by Theorem 1 we have $c^\top x_d \longrightarrow c^\top x^*$ as $d \longrightarrow 0$. To get the result of convergence rate, we need to check three assumptions (A_1), (A_2), and (A_3) introduced in above section for (P′).

(a) For the solution x^* of (P′), there is a neighborhood U of x^*, such that the function $D_y^2 g(x, y)$ is continuous on $U \times Y$ where $g(x, y) = y^\top F(x) y$.

(b) The index set $Y \subset R^n$ is compact, nonempty and explicitly given as the solution set of inequalities:

$$Y = \{y \in R^n \mid v_i(y) \le 0,\ i \in I\},$$

where $I = \{1, 2\}$ and $v_1(y) = \alpha - \|y\|^2 (0 < \alpha < 1)$ for a given α, $v_2(y) = \|y\|^2 - 1$.

(c) The *Mangasarian-Fromovitz Constraint Qualification* (MFCQ) holds for Y: for any $\bar{y} \in Y$ with the active set $I(\bar{y}) := \{i \in I \mid v_i(\bar{y}) = 0\}$ there exists a vector $\bar{\eta} = \eta(\bar{y})$, such that

$$Dv_i(\bar{y})\, \bar{\eta} < 0, \qquad i \in I(\bar{y}).$$

(a)–(c) are obvious. So A_1 holds. The conditions in theorem ensure that A_2 holds. Now we verify A_3, i.e., MFCQ (defined by (A_3) in Section 2) is valid for x^*.

In fact, $D_x g(x, \bar{y}) = [\bar{y}^\top F_1 \bar{y}, \dots, \bar{y}^\top F_m \bar{y}]^\top$, since (P) is strictly feasible there exists an $\tilde{x}$, such that $F_0 + \sum_{i=1}^m \tilde{x}_i F_i \succ 0$, i.e., $y^\top (F_0 + \sum_{i=1}^m \tilde{x}_i F_i) y > 0$, for all $y \in Y$. Let $\beta = \min_{y \in Y} y^\top (F_0 + \sum_{i=1}^m \tilde{x}_i F_i) y (> 0)$ and $\xi = (x^* - \tilde{x})/\beta$. Since

$$\bar{y}^\top F_0 \bar{y} + \bar{y}^\top \left(\sum_{i=1}^m x^* F_i \right) \bar{y} = 0,$$

we have

$$D_x g(x^*, \bar{y})\, \xi = -\frac{\bar{y}^\top \left(F_0 + \sum_{i=1}^m \tilde{x}_i F_i \right) \bar{y}}{\beta} \le -1, \qquad \text{for all } \bar{y} \in Y_0(x^*).$$

So A_3 holds.

Notice the fact that (P′) is equivalent to (P), and by Lemma 3 we obtain the final result (3.2). ∎

3.2. Duality

In this section, with the conclusions we have got with the discretization method for (SDP), we give a simple proof of the duality result in semi-definite programming.

The associated dual problem of problem (P) is

$$\begin{aligned} &\max && -\operatorname{Tr} F_0 Z, \\ &\text{subject to} && \operatorname{Tr} F_i Z = c_i, \qquad i = 1, \dots, m, \\ &&& Z \succeq 0, \end{aligned} \tag{D)(3.3}$$

where Tr means the trace of a matrix.

Let val (P) be the optimal value of problem (P), val (D) be the optimal value of problem (D). Then the following duality result holds.

THEOREM 3. *(See [1].) If (P) and (D) are strictly feasible, then* val (P) = val (D).

PROOF. First, we give the dual problem of (P_d).

For $Y_d = \{y_1, y_2, \dots, y_l\}$, (P′) is a linear optimization problem with l constraints, i.e.,

$$\begin{aligned} &\min && c^\top x, \\ &\text{subject to} && Ax \le b, \end{aligned}$$

where

$$A = -\begin{pmatrix} y_1^\top F_1 y_1 & \cdots y_1^\top F_m y_1 \\ \vdots & \ddots \vdots \\ y_l^\top F_1 y_l & \cdots y_l^\top F_m y_l \end{pmatrix}, \qquad b = \begin{pmatrix} y_1^\top F_0 y_1 \\ \vdots \\ y_l^\top F_0 y_l \end{pmatrix} \tag{3.4}$$

With the result of linear programming, we get the dual of (P_d):

$$\begin{aligned} &\max && -\operatorname{Tr} F_0 Z \\ &\text{subject to} && \operatorname{Tr} F_i Z = c_i, \qquad i = 1, \dots, m, \\ &&& Z \in Z_d, \end{aligned} \tag{D$_d$) (3.5}$$

where $Z_d = \{Z \mid Z = \sum_{j=1}^{l} t_j y_j y_j^\top,\ t \ge 0\}$. Let val ($P_d$) be the optimal value of problem (P_d), and val (D_d) be the optimal value of the problem (D_d). For every feasible x and Z to (P) and (D), we have

$$c^\top x - (-\operatorname{Tr} F_0 Z) = \operatorname{Tr}\left(F_0 + \sum_{i=1}^{m} x_i F_i\right) Z \ge 0. \tag{3.6}$$

Therefore,

$$\operatorname{val}(P) \ge \operatorname{val}(D).$$

Since (D) is feasible, val (P) is finite. Let x^* be an optimal solution of (P). As we discussed above, we can generate subproblem (P_d), such that $x_d \longrightarrow x^*$ as $d \longrightarrow 0$. Since the feasible set of (P) is included in the feasible set of (P_d), then

$$\operatorname{val}(\mathrm{P}_d) \le \operatorname{val}(\mathrm{P}). \tag{3.7}$$

At the same time, by using Theorem 2 we have val (P_d) $\longrightarrow$ val (P) as $d \longrightarrow 0$. With the finiteness of val (P), we can assume that d is small enough, such that val (P_d) is bounded. Now with linear programming theory we have val (P_d) = val (D_d). While the feasible set of (D_d) is included in the feasible set of (D), thus

$$\operatorname{val}(\mathrm{D}_d) \le \operatorname{val}(\mathrm{D}). \tag{3.8}$$

So with d small enough, we can guarantee that

$$\operatorname{val}(\mathrm{P}_d) = \operatorname{val}(\mathrm{D}_d), \qquad \operatorname{val}(\mathrm{D}_d) \le \operatorname{val}(\mathrm{D}),$$

Let d approach zero, one has val (P) $\le$ val (D).

It follows that val (P) = val (D) since val (P) $\ge$ val (D) from above. ∎

COROLLARY 1. *Under the hypothesis of Theorem 2, there is some $\gamma_4 > 0$, such that for d small enough,*

$$0 \leq \mathrm{val}\,(\mathrm{D}) - \mathrm{val}\,(\mathrm{D}_d) \leq \gamma_4\, d^2. \tag{3.9}$$

It is obvious with Theorems 2 and 3.

Note that the discretization problems (P_d) and (D_d) converge to problems (P) and (D), respectively, at the rate of $O(d^2)$, we can apply some polynomial time methods to solve (P_d) or (D_d) and get the approximal value of (P) or (D). The scale of (P_d) and (D_d) is related to the dimension of Y and the value of d, and will be fairly large in practice. However our analysis indicates that the discretization method is an efficient method to handle the semi-definite problem.

4. A DISCUSSION ABOUT THE OPTIMAL DIVISION

In this section, we discuss how to get approximatively optimal division Y_d for a given tolerance $\epsilon > 0$.

First, we analyze the computing complexity to solve (D_d).

In practice, we can construct a semi-infinite problem with $E = [-1,1]^n$ substituting for Y. It doesn't destroy the equivalence of (P′) with (P). We divide every interval $[-1,1]$ into $(2h+1)$ (h is a positive integer) segments with the nodes denoted as $-1 + (2i/2h+1)$, $i = 0, 1, \ldots, 2h+1$. Let E_d be the grid point set of E. It is easily verified that the hypothesis in Theorem 2 are met. With the division, we can calculate the number of all the grid points in E_d as $l = (2h+2)^n$. In this instance, the Hausdorff distance is

$$d = \frac{2\sqrt{n}}{2h+1} = \frac{2\sqrt{n}}{l^{1/n}-1}. \tag{4.1}$$

Since (D_d) is a standard linear programming, it can be solved with polynomial time interior-point methods. For convenience, we write (D_d) as follows:

$$\begin{aligned} \min\ & b^\top v \\ \text{s.t.}\ & A^\top v = c, \\ & v \geq 0, \end{aligned} \tag{4.2}$$

where A and b are defined as (3.4). Here we assume that we solve (D_d) with primal Karmarkar interior-point algorithm. Initially, we need $(m+1)(2n^2+n-1)l$ times of computation to establish the coefficients of (D_d). Let v_{d_k} be the solution generated by Karmarkar method with k iteration steps, x_d, v_d, and x^* are the solutions of (P_d), (D_d), and (P), respectively. Based on the analysis of the Karmarkar method (see [6] for instance), we have the estimation

$$\left|b^\top v_{d_k} - c^\top x_d\right| = \left|b^\top v_{d_k} - b^\top v_d\right| \leq e^{-k/5l} b^\top e/l. \tag{4.3}$$

Here we assume that Karmarkar's standard form has been obtained and e is the l-dimension vector of ones. For following discussion, we assume $b^\top e/l \leq 1$ which implies $\sum_{i=1}^{l} y_i^\top F_0 y_i \leq l$.

Then, for a given ϵ_1, if $k \geq \lfloor -5l \ln \epsilon_1 \rfloor + 1$ (here we use $\lfloor a \rfloor$ to express the largest integer less than or equal to a), then

$$\left|b^\top v_{d_k} - c^\top x_d\right| \leq \epsilon_1.$$

Based on the Theorem 2 in Section 3, we also know that when d is sufficiently small, $|c^\top x_d - c^\top x^*| \leq \gamma_3 d^2$. And for a given ϵ_2, if $\gamma_3 d^2 = 4\gamma_3 n/(l^{1/n}-1)^2 \leq \epsilon_2$, we have

$$\left|c^\top x_d - c^\top x^*\right| \leq \epsilon_2.$$

For a given ϵ, let $\epsilon_1 = \delta\epsilon$, $\epsilon_2 = (1-\delta)\epsilon$, where $0 < \delta < 1$. With the above analysis, we adopt $k = \lfloor -5l \ln \epsilon_1 \rfloor + 1$ and $l = \lfloor \{2\sqrt{\gamma_3 n/(1-\delta)\epsilon} + 1\}^n \rfloor + 1$ which implies that $\gamma_3 d^2 < (1-\delta)\epsilon$. So we guarantee that

$$\left|b^\top v_{d_k} - c^\top x^*\right| \leq \left|b^\top v_{d_k} - c^\top x_d\right| + \left|c^\top x_d - c^\top x^*\right| \leq \epsilon. \tag{4.4}$$

Since it is sufficient to take $(2/3)\, l^3$ times operations at each iterate k to get d_k with Karmarkar method for large l, we get the total computing complexity estimation:

$$C(l, \delta) = \frac{2}{3}\, l^3 k + (m+1)\left(2n^2 + n - 1\right) l. \tag{4.5}$$

Furthermore, we approximate $C(l, \delta)$ with

$$\begin{aligned} F(\delta) = \tilde{C}(l, \delta) &= -\frac{10}{3}\left\{2\sqrt{\frac{\gamma_3 n}{(1-\delta)\epsilon}} + 1\right\}^{4n} (\ln\delta + \ln\epsilon) \\ &+\frac{2}{3}\left\{2\sqrt{\frac{\gamma_3 n}{(1-\delta)\epsilon}} + 1\right\}^{3n} + (m+1)\left(4n^2 - 2n\right)\left\{2\sqrt{\frac{\gamma_3 n}{(1-\delta)\epsilon}} + 1\right\}^{n}. \end{aligned} \tag{4.6}$$

The approximately optimal division of E with the minimal computation is determined with the following optimal problem:

$$\min_{0<\delta<1} F(\delta). \tag{4.7}$$

With the first-order optimal condition, we can get the approximately optimal solution δ of (4.6). So we approximately know how to divide E, and how many iteration steps in solving (D_d). Then the approximately minimal number of grids is determined since $l = \lfloor\{2\sqrt{\gamma_3 n/(1-\delta)\epsilon} + 1\}^n\rfloor + 1$.

REMARKS.

1. The above analysis to estimate the approximately optimal division can be applied to semi-definite programming with the discretization solution method while using other polynomial time algorithms to solve (D_d).
2. Since the division of E is regular, we can apply parallel computing method to get the coefficients of (D_d). It may approve the computing efficiency. Furthermore, if l is very large, we can apply the technique introduced in [3] to eliminate some constraints. So the scale of (D_d) is reduced.
3. Since (P') is still equivalent to (P) if we take

   $$Y = \{y \mid -1 \le y_i \le 1, 1 \le i \le n, \text{ and } \exists\, |y_j| = 1\}.$$

   We can also solve (D_d) only using the grid nodes on the boundary of E. In this way, the number of useful grid nodes reduces to $(2h+2)^n - (2h)^n$. So we can get required solution with less cost.

5. CONCLUSIONS

In this paper, we reformulate the semi-definite programming (P) as semi-infinite programming (P') which can be solved with the discretization method. So we generalize the discretization method to the semi-definite programming problems. Moreover, we prove that the optimal value of the discretized problem (P_d) converges to the optimal value of the (SDP) problem (P) at the rate of $O(d^2)$. Applying the discretization method, we avoid the difficulty to judge whether the condition $F(x) \succeq 0$ holds. While to solve every subproblem (P_d), we must deal with fairly many constraints. Our analysis in Section 4 indicates that for a given ϵ, we can get an optimal δ which approximatively determines the division of Y, such that the computing complexity degree is approximatively minimal. Furthermore, we refer to [3] for some skills in solving the discretized problem. Another contribution in this paper is the simple proof for part of the strong duality theory in semi-definite programming.

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